On the Co-orbital Motion of Two Planets in Quasi-circular and Co-planar Orbits Focused on the Anti-Lagrange Orbits

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Introduction

The co-orbital resonance has been extensively studied for more than one hundred years in the framework of the restricted three-body problem (RTBP). In most of the analytical works, the emphasis has been placed on the tadpole orbits since these describe the motion of the Jovian Trojans.

[Giuppone et al., 2010] studied the stability regions and families of periodic orbits of two planets locked in the co-orbital resonance with a numerical averaging of the disturbing function. Besides the Lagrangian triangular configurations, these authors found a new family of fixed points that they called anti-Lagrange orbits and caracterized by the relation $m_1e_1 \simeq m_2e_2$.

This poster and its associated paper ([Robutel and Pousse, 2013]) show that these Lagrange and Anti-Lagrange families bifurcate along the eccentricities direction with the same origin: the Lagrangian triangular configuration.

The averaged Hamiltonian

We consider two planets of masses m_1 and m_2 respectively in a co-planar motion, a central body (Sun, or star) of dominant mass m_0 with respect to the planetary masses. Following [Laskar and Robutel, 1995], the Hamiltonian of the three-body problem reads

$$H(\tilde{\mathbf{r}}_j,\mathbf{r}_j) = \underbrace{\sum_{j\in\{1,2\}}\left(\frac{\tilde{\mathbf{r}}_j^2}{2\beta_j} - \frac{\mu_j\beta_j}{||\mathbf{r}_j||}\right)}_{H_K(\varepsilon)} + \underbrace{\frac{\tilde{\mathbf{r}}_1\cdot\tilde{\mathbf{r}}_2}{m_0} - \mathcal{G}\frac{m_1m_2}{||\mathbf{r}_1-\mathbf{r}_2||}}_{H_p(\varepsilon^2)}, \quad \text{where} \tag{1}$$

- $\mathbf{r}_j$ is the heliocentric position of the planet j and $\tilde{\mathbf{r}}_j$, its conjugated variable, is the barycentric linear momentum of this body,
- $\beta_j = m_0m_j(m_0+m_j)^{-1}$ and $\mu_j = \mathcal{G}(m_0+m_j)$, $\mathcal{G}$ being the gravitational constant,
- H_K corresponds to the unperturbed Keplerian motion of the two planets (motion of a mass β_j around a fixed center of mass m_0+m_j),
- H_p models the gravitational perturbations,
- $\varepsilon = \text{Max}\left(\frac{m_1}{m_0},\frac{m_2}{m_0}\right)$ is a small parameter.

$\Rightarrow H_K$ **is of order** ε **and** H_p **is of order** ε^2**: that justifies a perturbative approach.**

As we only consider the planetary motions in a plane and in the vicinity of the circular problem, the Hamiltonian can be expanded in power series of the eccentricity variables and their conjugates in the form

$$\sum_{k_1,k_2}\left(\sum_{(\mathbf{p},\mathbf{q})\in\mathbb{N}^8}\Psi^{k_1,k_2}_{\mathbf{p},\mathbf{q}}(\Lambda_1,\Lambda_2)X_1^{p_1}X_2^{p_2}\overline{X}_1^{\bar{p}_1}\overline{X}_2^{\bar{p}_2}\right)e^{i(k_1\lambda_1+k_2\lambda_2)} \quad \text{with} \quad X_j, \quad \text{equivalent to} \quad e_j\exp(i\varpi_j) \quad \text{for} \quad e_j\sim 0, \tag{2}$$

the non dimensional quatity associated to the Poincaré variables x_j.

Remember that $(\lambda_j,\Lambda_j,x_j,-i\overline{x}_j)$, the Poincaré rectangular variables in complex form are a canonical coordinate system related to the elliptical elements $(a_j,e_j,\lambda_j,\varpi_j)$.

$\Rightarrow$**The angular momentum of the system is an integral of the motion ="D'Alembert rule" which implies** $\sum_j(k_j+p_j-\bar{p}_j)=0$.

In order to deal with the co-orbital resonance, we choose a more appropriate canonical coordinate system with

$$\begin{aligned}\theta_1 &= \lambda_1-\lambda_2, & 2J_1 &= \Lambda_1-\Lambda_2,\\ \theta_2 &= \lambda_1+\lambda_2, & 2J_2 &= \Lambda_1+\Lambda_2.\end{aligned} \tag{3}$$

You can notice that inside the 1:1 mean motion resonance, θ_1 varies slowly with respect to θ_2 . Consequently, **the planetary Hamiltonian (1) will be averaged over the angle** θ_2.

Then, at first order in the planetary masses, the Hamiltonian becomes $\overline{H}(\theta_j,J_j,x_j,-i\overline{x}_j) = \overline{H}_0(J_j) + \overline{H}_1(\theta_1,J_j,x_j,-i\overline{x}_j) + \mathcal{O}(\varepsilon^3)$ with

$$\overline{H}_0(J_1,J_2) = -\frac{\beta_1^3\mu_1^2}{2(J_1+J_2)^2} - \frac{\beta_2^3\mu_2^2}{2(J_1-J_2)^2} = H_K\circ\phi(\theta_j,J_j,x_j,-i\overline{x}_j) \quad \text{and} \quad \overline{H}_1(\theta_1,J_j,x_j,-i\overline{x}_j) = \frac{1}{2\pi}\int_0^{2\pi}H_p\circ\phi(\theta_j,J_j,x_j,-i\overline{x}_j)d\theta_2, \tag{4}$$

where the map ϕ satisfies the relation $(\tilde{\mathbf{r}}_j,\mathbf{r}_j) = \phi(\theta_j,J_j,x_j,-i\overline{x}_j)$.

$\Rightarrow$ **The averaging implies that** $\overline{H}$ **is independent of** θ_2.

This fact has two main consequences :

- J_2 is a first integral,
- D'Alembert rule imposes that $\overline{H}$ is even in the variables x_j and their conjugates.

This last property implies that **the set** $\{x_1=x_2=0\}$ **is an invariant manifold with respect to the flow** of the averaged Hamiltonian (4).

$\Rightarrow$ **The part of the averaged Hamiltonian (4) which doesn't depend on eccentricities (**$H_0(\theta_1,J_j) = \overline{H}(\theta_1,J_j,0,0)$**) is an integrable Hamiltonian**.

It is worth noting that the one degree of freedom Hamiltonian H_0, associated to the circular and planar resonant problem, is a peculiar attribute of the 1:1 mean-motion resonance.

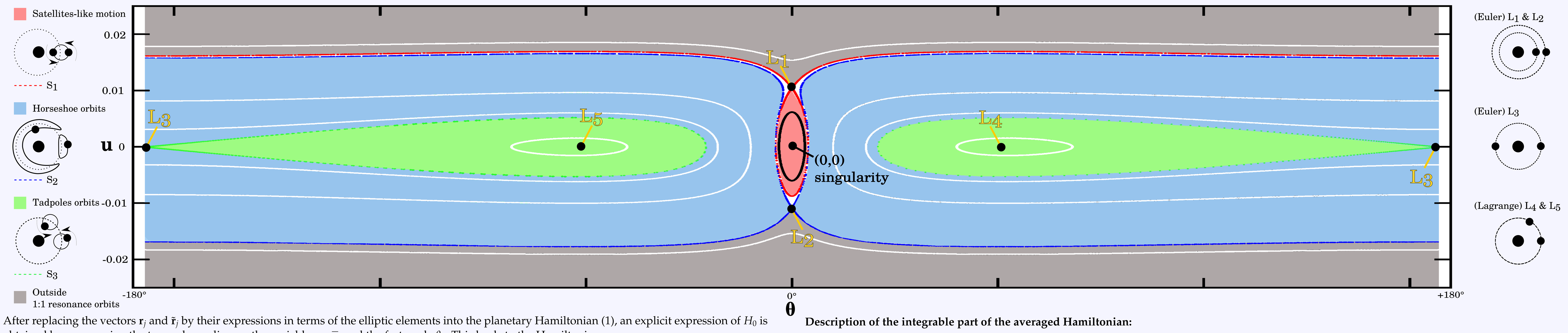
The first property allows us to define a new canonical coordinate system which inserts the stricly positive number $\overline{a}$ and such that

$$J_2 = \frac{\beta_1\sqrt{\mu_1}+\beta_2\sqrt{\mu_2}}{2}\sqrt{\overline{a}}, \quad J_1 = \frac{\beta_1\sqrt{\mu_1}-\beta_2\sqrt{\mu_2}}{2}\sqrt{\overline{a}}+J, \quad \theta_1=\theta. \tag{5}$$

Finally, we introduce the dimensionless and non canonical action-like variable u as $J\propto u$ and which can be relied to the semi-major axes : $\overline{a}u^2 \propto a_1-a_2$.

Thus, the integrable averaged Hamiltonian H_0 can be studied and explicitely expressed in terms of the $(\theta,J,\overline{a})$, or $(\theta,u,\overline{a})$ for convenience.

The integrable part H_0



After replacing the vectors $\mathbf{r}_j$ and $\tilde{\mathbf{r}}_j$ by their expressions in terms of the elliptic elements into the planetary Hamiltonian (1), an explicit expression of H_0 is obtained by suppressing the terms depending on the variables $x_j,\overline{x}_j$ and the fast angle θ_2. This leads to the Hamiltonian

$$H_0 = -\frac{\beta_1\mu_1}{2a_1} - \frac{\beta_2\mu_2}{2a_2} + \mathcal{G}m_1m_2\left(\frac{\cos\theta}{\sqrt{a_1a_2}} - \frac{1}{\sqrt{a_1^2+a_2^2-2a_1a_2\cos\theta}}\right). \tag{6}$$

The above figure represents the whole phase portrait of the integrable Hamiltonian H_0 in coordinates (θ,u) for $m_1=m_J=10^{-3}$ and $m_2=m_S=3\times10^{-4}$ and $\mathcal{G}=m_0=\overline{a}=1$, where the masses m_J and m_S are close to those of Jupiter and Saturn expressed in solar mass.

Firstly, you can notice that this plot is similar to the well known Hill's diagram (or zero-velocity curves) of the non averaged planar circular RTBP (see [Szebehely, 1967]) although the zero-velocity curves are not solution curves of the motion. It is also topologically equivalent to the phase space of the averaged planar circular RTBP when the eccentricity of the test-particle is equal to zero (see [Nesvorný et al., 2002, Morbidelli, 2002]).

Description of the integrable part of the averaged Hamiltonian:

- *Singularity :* H_0 possesses **one singular point at** $u=\theta=0$ which corresponds to the **collision** between the planets.
- *Equilibria & Dynamics :* This Hamiltonian possesses **five fixed points** that correspond to the usual **Euler and Lagrangian configurations** .
  - The **two stable equilibrium points located at** $\theta=\pm\pi/3, u=0$ represent the averaged equilateral configurations that we will denote abusively by L_4 **and** L_5 **by analogy with the RTBP**. Each of these points is surrounded by **tadpole orbits** corresponding to periodic deformations of the equilateral triangle. This region is bounded by the separatrix $\mathcal{S}_3$ that originates at the **hyperbolic fixed point** L_3 **at** $\theta=\pi, u\approx 0$, for which the three bodies are aligned and the Sun is between the two planets and its separatrix.
  - Outside this domain, **the horseshoe orbits** are enclosed by the separatrix $\mathcal{S}_2$ that originates at the fixed point L_2 ($\theta=0$ and $u<0$). This point, as the equilibrium point L_1, is associated with an Euler configuration for which the two planets are on the same side of the Sun.
  - The last domain, centered at the singularity, is surrounded by the separatrix $\mathcal{S}_1$ connecting the L_1 point ($\theta=0$ and $u>0$) to itself. Inside this small region, **the two planets seem to be subjected to a satellite-like motion**.

In addition to this, for equal planetary masses, with respect to the axis $u=0$ the phase portrait becomes symmetric . It turns out that the equilibrium points L_3,L_4,L_5 lie on the axis of symmetry, and that the two curves $\mathcal{S}_1$ and $\mathcal{S}_2$ merge together giving rise to a unique separatrix connecting L_1 to L_2.

Infinitesimal neighborhood of H_0

In order to study the linear stability of the invariant manifold in the eccentricities direction, we calculate the variational equations associated to this invariant surface.

$\Rightarrow$ **Linearization of the differential system associated to (4) in the neighborhood of** $\{x_j=0\}$.

It can be derived from the **quadratic expansion in eccentricity** of the averaged Hamiltonian $\overline{H}$. This expansion can be written in the form $H_0+H_2^{(h)}$ with

$$H_2^{(h)} = \mathcal{G}m_1m_2\left(A_hX_1\overline{X}_1 + B_hX_1\overline{X}_2 + \overline{B}_h\overline{X}_1X_2 + A_hX_2\overline{X}_2\right) \tag{7}$$

where A_h and B_h are $(\theta(t),u(t))$-dependent coefficients.

The variational equations in the vicinity of a solution lying in the plane $\{x_j=0\}$ and satisfying

$$\dot{\theta} = \frac{1}{c}\frac{\partial H_0}{\partial u}(\theta,u), \quad \dot{u} = -\frac{1}{c}\frac{\partial H_0}{\partial\theta}(\theta,u) \quad (c \text{ as dimensionnal constant) take the form} \tag{8}$$

$$\begin{pmatrix}\dot{X}_1\\ \dot{X}_2\end{pmatrix} = M_h(\theta,u)\begin{pmatrix}X_1\\ X_2\end{pmatrix} \quad \text{with} \quad M_h(\theta,u) = 2i\mathcal{G}m_1m_2\begin{pmatrix}\Lambda_1^{-1}A_h & \Lambda_1^{-1}\overline{B}_h\\ \Lambda_2^{-1}B_h & \Lambda_2^{-1}A_h\end{pmatrix} \tag{9}$$

As these solutions are periodic (except if their initial conditions are chosen on separatrices) the linear equation (9) is periodically time-dependent. As a consequence, their solutions cannot generally be expressed in a closed form.

$\Rightarrow$ **A notable exception occurs at the equilibrium points of the system (8) where the variational equations become autonomous and consequently integrable.**

Thus, we study the (linear) neighborhood along eccentricity direction of the circular Lagrangian equilibria L_4 and L_5.

The emergence of anti-Lagrange orbits

For the equilateral configurations ($\theta=\pm\pi/3$), neglecting the quadratic terms in ε, the matrix M_h takes the following expression

$$M_h = -i\frac{27}{8}\frac{n_0}{m_0}\begin{pmatrix}m_2 & -m_2e^{i\theta}\\ -m_1e^{-i\theta} & m_1\end{pmatrix} \quad \text{where } n_0 \text{ plays the role of an averaged mean motion} \tag{10}$$

Matrix study: This matrix possesses two eigendirections associated to the eigenvectors

$$V_1 = \begin{pmatrix}e^{i\theta}m_2\\ -m_1\end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix}e^{i\theta}\\ 1\end{pmatrix} \quad \text{whose eigenvalues are respectively} \quad \begin{cases}v_1 = -i\frac{27}{8}\frac{m_1+m_2}{m_0}n_0\\ v_2 = 0\end{cases} \tag{11}$$

Physical interpretation:

- V_2 direction: Along the neutral direction, the one which is collinear with V_2, the two eccentricities are the same and the angle $\Delta\varpi = \varpi_1-\varpi_2$ separating the two apsidal lines is equal to $\pi/3$ at L_4 and $-\pi/3$ at L_5. These configurations clearly correspond to the **Lagrangian elliptic equilibria**, which are fixed points of the averaged problem, and consequently of the linearized averaged problem at L_4 or L_5. This is the reason why the associated eigenvalue v_2 vanishes.
- V_1 direction: Along the direction V_1, the orbits satisfy the relations

$$a_1=a_2=\overline{a}, \quad \theta=\pm\pi/3, \quad m_1e_1=m_2e_2, \quad \text{and} \quad \Delta\varpi=\varpi_1-\varpi_2=\theta+\pi.$$

This corresponds to **an infinitesimal version of the anti-Lagrange orbits** found numerically by [Giuppone et al., 2010]. On these trajectories the elliptic elements a_1,a_2,e_1,e_2 and θ are constant. Only the two angles ϖ_1 and ϖ_2 precess with the same frequency equal to $g_1 = iv_1 = \frac{27}{8}\frac{m_1+m_2}{m_0}n_0$, in such a way that the angle $\Delta\varpi$ is constant.

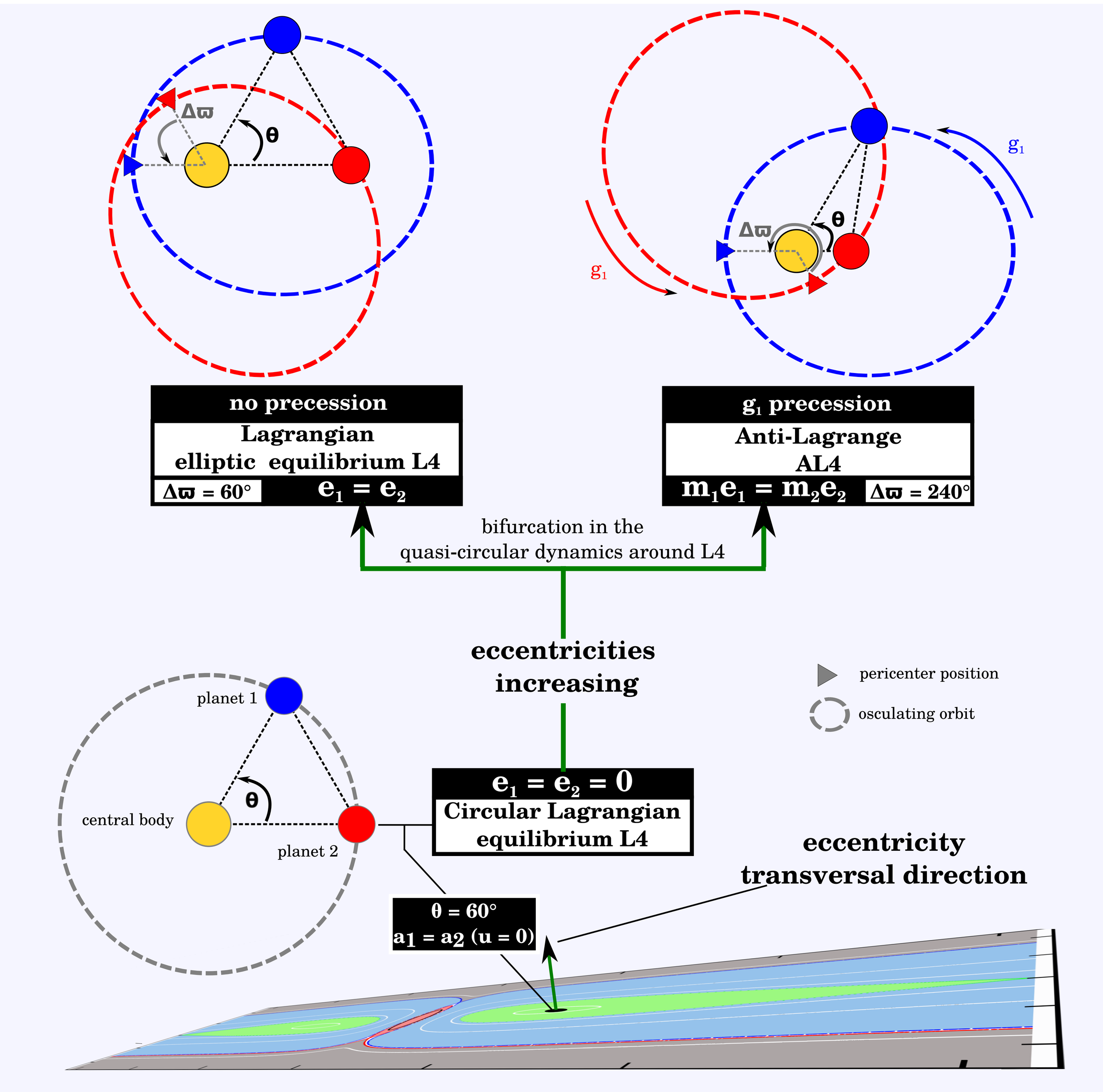
Of course, this family provides only an infinitesimal approximation of the anti-Lagrange family in the neighborhood of L_4 and L_5.

Generalization at any degree of $\overline{H}$

In this section, we describe how we proceeded in our paper [Robutel and Pousse, 2013] to generalize the infinitesimal approximation of the anti-Lagrange orbits discovered previously in a family of quasi-periodic orbits at any degree of the averaged Hamiltonian $\overline{H}$. The main idea was to use of the Birkhoff normal form transformation. In few words, we

- reduced the quadratic form to an Hamiltonian of a triple harmonic oscillators system and defined three families associated to each of them:
  - $\mathcal{F}_0$, the family that corresponds to the quasi-circular motions ($e_1=e_2=0$),
  - $\mathcal{F}_1^l$, the one corresponding to the linear approximation of the anti-Lagrange orbits,
  - $\mathcal{F}_2$, the last one which contains the Lagrangian elliptic configurations.
- Considering the terms of degree greater than two in the expansion of the averaged Hamiltonian, we noticed that the D'Alembert rule holds, implying that the manifold $\{x_j=0\}$ is still invariant to the flow of this Hamiltonian. Hence, $\mathcal{F}_0$ and $\mathcal{F}_2$ are invariant of the full averaged Hamiltonian also.
- Then, we constructed a coordinate system for which the anti-Lagrange family possesses the same kind of parametrization than the two other families (just depending on one element which characterizes the entire family). This coordinate system can be chosen among one of those that reduce the full averaged Hamiltonian to its Birkhoff normal form.

In the validity domain of the Birkhoff normal form, we established rigorously the existence of the family $\mathcal{F}_1$ **which satisfies the relation** $m_1e_1=m_2e_2$ **at its beginning. We found analytically the anti-Lagrange orbits which were deduced by numerical simulations in [Giuppone et al., 2010]. Thus, the elliptic Lagrangian configurations and the anti-Lagrange orbits bifurcate along eccentricities direction from the same fixed point: the circular Lagrangian equilibria** L_4 **and** L_5**.**



Scheme of the dynamics along the eccentricities direction on the fixed point L_4 at the infinitesimal state. The elliptic Lagrangian equilateral configurations and the Anti-Lagrange orbits bifurcate at the same fixed point L_4.

Conclusion

We developed an analytical Hamiltonian formalism adapted to the study of two planets in a co-planar motion and co-orbital resonance. The Hamiltonian, averaged over one of the planetary mean longitude, is expanded in power series of eccentricities. The model, which is valid in the entire co-orbital region, possesses an integrable approximation modeling the planar and quasi-circular motions. Focusing on the two Lagrangian fixed points of this approximation, we highlighted relations linking the eigenvectors of the associated linearized differential system and the existence of remarkable orbits like the elliptic Lagrangian configurations and the anti-Lagrange orbits ([Giuppone et al., 2010]). The bottom figure summerizes what we found with the help of the construction of a Birkhoff normal form: the circular Lagrangian equilibria (L_4 **and** L_5**) bifurcates along eccentricities direction to the elliptic Lagrangian equilateral configurations and the anti-Lagrange orbits.**

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