On Co-orbital Motion in the Coplanar Restricted Three-body Problem
Quasi-Satellites in the Circular Case

Alexandre Pousse[(1)], Philippe Robutel[(1)], Alain Vienne[(1)]

apousse@imcce.fr robutel@imcce.fr vienne@imcce.fr

[(1)] IMCCE - Observatoire de Paris, Université Lille 1, Université Paris 06

l'Observatoire de Paris — Institut de Mécanique Céleste et de Calcul des Éphémérides

COMPLEX PLANETARY SYSTEMS • IAU-SYMPOSIUM

I. Introduction

- In the framework of a perturbative treatment of the Restricted Three-body Problem (RTBP), we consider an asteroid as a particle orbiting around a star, perturbated by the gravitationnal influence of a planet, in the configuration of co-orbital resonance.

  We actually know three classes of regular co-orbital motions: in the rotating frame with the planet,
  - the **tadpoles (TP)** librate around Lagrangian equilibria L_4 or L_5,
  - the **horseshoe orbits (HS)** encompass the three equilibrium points L_3 , L_4 and L_5,
  - **the quasi-satellites orbits (QS) seem to be a remote retrograde satellite in the rotating planetocentric frame, while it is instead on an elliptical heliocentric motion as it stands outside the influence sphere of the planet itself.**
- We can reformulate QS motion definition by elliptical heliocentric orbits which librate in the rotating frame around the center of libration located on the planet position. Instead of TP, QS do not emerge from a fixed point in the rotating frame as they always are on eccentric orbits.

  The goal of the poster is to understand how the QS domain emerges by studying the phase space of the coplanar RTBP with a planet in circular motion (CRTBPC).

II. Phase space in the circular case of the coplanar averaged RTBP ($\overline{\text{CRTBPC}}$)

- We define a perturbative parameter ε which caracterizes the Star-Planet system and represents the ratio mass planet over sum of the masses.
- Assuming that we study orbits far from planet close encounters, we can work in the approximation of the averaged problem over the planet revolution (noted $\overline{\text{CRTBPC}}$) where we consider at a time t the mean of the gravitationnal influence of the planet over a revolution instead of its real influence value. The phase space of the $\overline{\text{CRTBPC}}$ can be represented in terms of this mean elements:
  - $\boldsymbol{\theta = \lambda - \lambda'}$, resonant angle corresponding to the difference of the mean longitudes,
  - $\boldsymbol{u} = \frac{\sqrt{a}-\sqrt{a'}}{\sqrt{a'}}$ with a and a' semi-major axes of the asteroid and the planet, vicinity to the resonance,
  - $\boldsymbol{e}$, eccentricity of the asteroid orbit,
  - $\boldsymbol{\omega}$, argument of the periastre of the asteroid orbit.

  An important consequence of studying $\overline{\text{CRTBPC}}$ is the reduction of dimensions: from now, we have only two degrees of freedom.
- Moreover there is an invariant symmetry by rotation giving the first integral $\boldsymbol{\Gamma = \sqrt{a}(1-\sqrt{1-e^2})}$ and allowing us to cut out the co-orbital phase space in Γ-sections, represented by planes in (θ, u).

  Remark: instead of using Γ, we define a more practical variable $\boldsymbol{e_0}$ which corresponds to e when $u=0$. If u is negligeable compared to 1, e_0 is closed to e.
- We use the method developped by Nesvorny et al. (2002) to compute the averaged Hamiltonian and equations of motions.

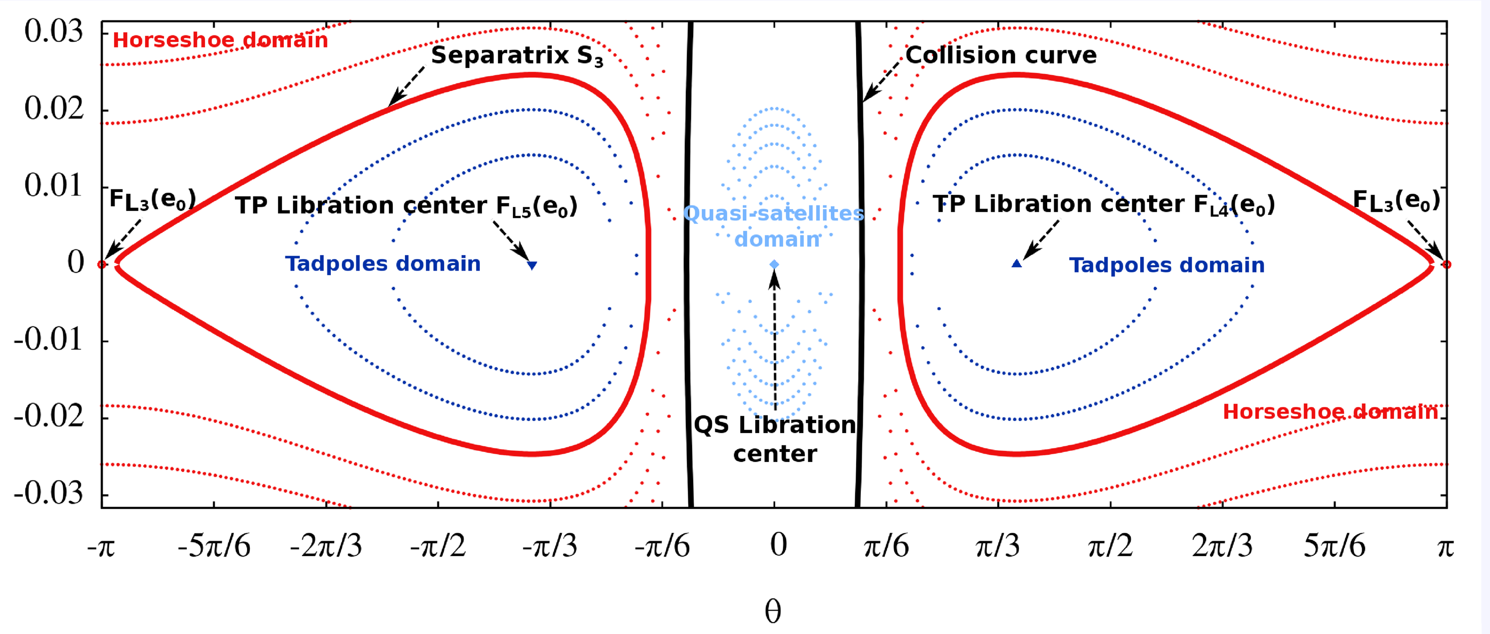

Fig.1: Section of the phase space of the $\overline{\text{CRTBPC}}$ for $e_0 = 0.2$ with a Sun-Jupiter system

- For a fixed value of e_0, libration centers' position in the rotating frame is assimilated to a fixed point with a libration frequency $\boldsymbol{\nu}$. But do not forget the additional degree of freedom given by ω and its associate precession frequency $\boldsymbol{g}$ –how the ellipse precesses in a fixed frame–.
- The representation of these fixed points in the non averaged problem are quasi-periodic orbits with two frenquencies (n and g) in the fixed frame. But in the rotating frame, they correspond to one frequency $(n-g)$ periodic orbits which seem satellized around a point (L_4 for TP with $e_0 = 0$, the planet for QS); along e_0, we obtain a family of periodic orbits where quasi-periodic orbits of QS or TP emerge.
  - $\mathcal{F}_{L_4}$ and $\mathcal{F}_{L_5}$ are the families of TP periodic orbits raised from the Lagrangian equilibria ($e_0 = 0$) from which TP quasi-periodic orbits emerge. In this plane, their libration center is located in the neighbourhood of their value in $e_0 = 0$: near $(\pm\frac{\pi}{3}, 0)$.
  - $\mathcal{F}_{L_3}$ is the unstable family of periodic orbits that originates from the unstable point L_3. In this plane, it represents an unstable point located near $(\pi, 0)$.
  - For $e_0 = 0.2$, the QS libration center is located near $(0,0)$.

III. – $\overline{\text{CRTBPC}}$ Study– Evolution of the QS libration center and associated frequencies

- As the QS libration center is seen as a fixed point, we use a numerical method –based on Newton-Raphson method– on the equations of motion of the averaged problem, to look for fixed point in the region of $(0,0)$ along e_0 and calculate the two frequencies values. To compare with TP and HS, we do the same study for $\mathcal{F}_{L_4}$, $\mathcal{F}_{L_5}$ and $\mathcal{F}_{L_3}$.
- We choose a Sun-Jupiter system ($\varepsilon = 0.001$).

IV. – $\overline{\text{CRTBPC}}$ Results– Averaged problem limit for QS motion and associated bifurcation

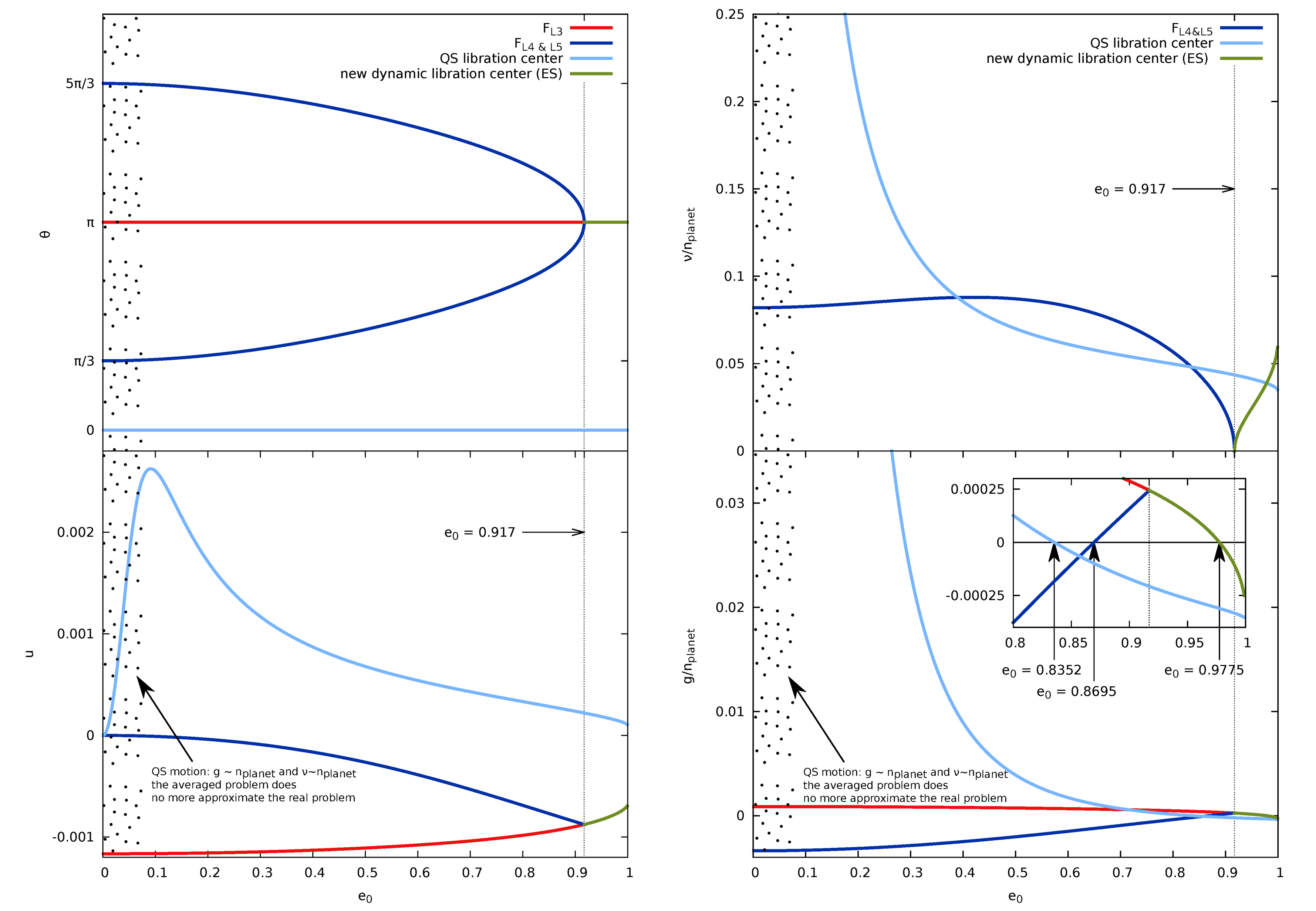

Fig.2: Fixed points evolution in (θ, u) plane and frequencies values along e_0 for a Sun-Jupiter system

- *On the position and frequencies evolution of the QS libration center:*
  - we note a limited domain where the averaged problem approximate the real problem: when $e_0 \lessapprox 0.1$, the precession and the libration evolve with a higher frequency than the mean mouvement n of the asteroid; due to the close encounters, the frequencies ν and g become of the same order that n. Consequently, although the average problem is well defined in this region, it does not represent the real problem
  - When $e_0 = 0.8352$, we remark the existence of a periodic orbit with the frequency precession crossing zero : we discover a frozen ellipse in QS motion.
- *On the bifurcation of the families originating from Lagragian points:*
  - Lagrangian equilibria's families evolve in the direction of $\theta = \pi$ and $u = O(\varepsilon)$: in the rotating frame, the TP libration center moves in the direction of the Lagrangian point L_3.
  - After $e_0 = 0.917$, the three families $\mathcal{F}_{L_3}$, $\mathcal{F}_{L_4}$ $\mathcal{F}_{L_5}$ merge and create a new family of stable fixed points in each plane (θ, u). This periodic orbits family orbiting around the point in opposition of the planet on the Star-Planet axis – that remind the "equant" point from the Ptolemea epicycling theory– span a new class of motion. For this reason, we suggest to call this new dynamic the "Equant Satellites" in comparaison to QS which seem orbiting around the planet (the equant satellites seem satellized around the equant point).

V. – $\overline{\text{CRTBPC}}$ and Rotating Frame Results – Phase space evolution (extension of the QS domain between the collision curve and bifurcation) and periodic orbits families evolution

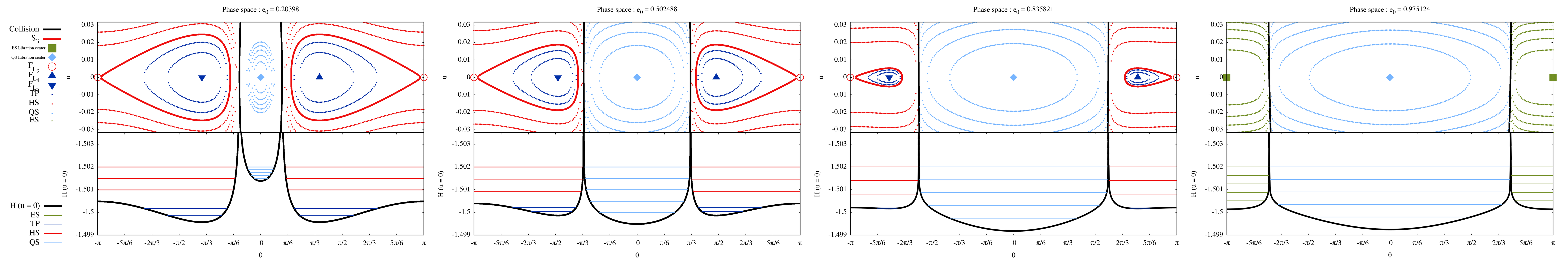

Fig.3: Evolution of the phase space and Hamiltonian value along e_0

Along e_0, the QS domain is always stuck between the collision curve and extend to very large value of θ. Moreover, the periodic family becomes a maximum of the Hamiltonian, implying that QS domain is a very stable domain for very high eccentricities. TP migrate towards $\theta = \pi$, and and merges with the others families. Then a new domain, caractirized by libration around π, appears: the equant satellites (ES).

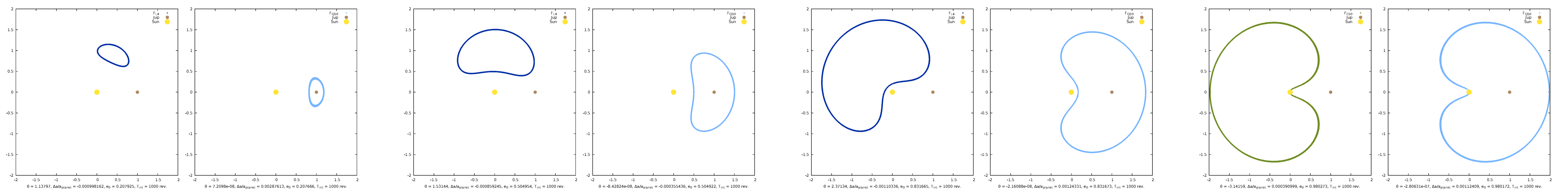

Fig.4: Numerical integrations after 1000 revolutions of QS, TP and also ES periodic orbits families conditions from the $\overline{\text{CRTBPC}}$ in the rotating frame of the CRTBPC

We confirm our results from the $\overline{\text{CRTBPC}}$ in the rotating frame of the non averaged problem: (QS) we find a family of symmetrical periodic orbit around the planet position; (TP) the libration center of TP evolve in direction to L_3; (ES) we find a new family of symmetrical periodic orbit that seems satellized around the "equant position". After the bifurcation value, we constate that ES and QS periodic orbits families are symmetrical and they seem to end with a collision with the Sun.

VI. – Results in Fixed Frame – the QS frozen orbit

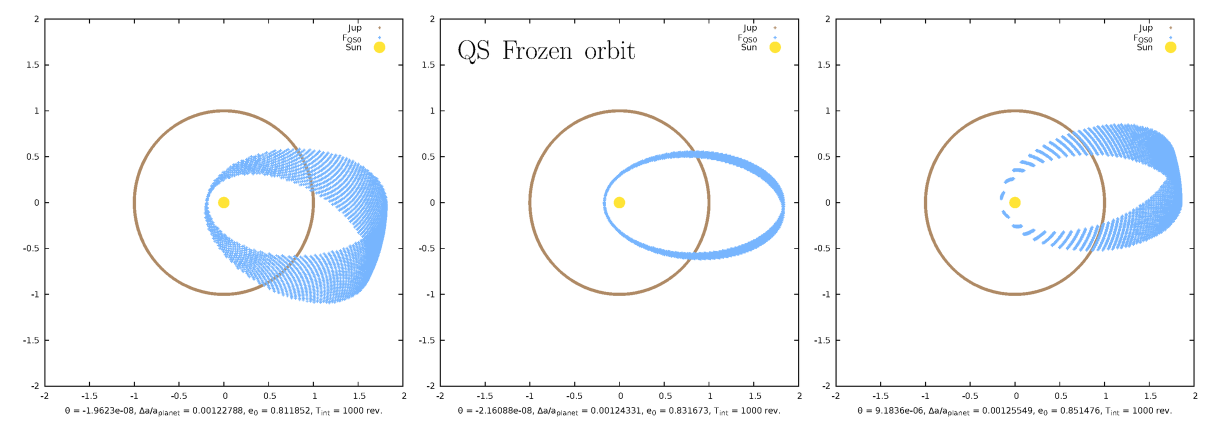

Fig.5: Numerical integration after 1000 revolutions of QS periodic orbits family conditions from the $\overline{\text{CRTBPC}}$ in a fixed frame of the CRTBPC, near the QS frozen orbit

For very high eccentricity ($e = 0.8352$), we confirm the existence of a QS orbit without ellipse precession which seems to freeze in the fixed frame despite of the planet perturbation.

VII. Conclusion

- *On QS motion*: we see that QS emerge from a two frequencies periodic orbits family which seems satellized around the planet position in the rotating frame. However, all the family is not available in the $\overline{\text{CRTBPC}}$ because of the close encounters when the eccentricity is low. To understand the origin of the family, you must refer to works of Jackson (1913) and Henon (1969) who showed the existence of a stable symmetrical periodic orbits family of retrograde satellites in CRTBPC in the rotating frame. We can conclude that QS motion emerge from a family of periodic orbits originating from planet collision, which corresponds to retrograde satellites orbits and then become elliptical heliocentric orbits which seems satellized around the planet, ending with a collision orbit with the Sun. We also showed the existence of a frozen ellipse condition at very high eccentricity ($e_0 = 0.8352$ for a Sun-Jupiter system).
- *On ES motion*: we discovered a new class of motion at very high eccentricity ($e_0 = 0.917$): the equant satellites. This new domain emerges from a family originating from the merging of the $\mathcal{F}_{L_4}$, $\mathcal{F}_{L_5}$ and $\mathcal{F}_{L_3}$ families and which end with the collision with the Sun. This implies the desappearance of TP and HS while a new domain of librating orbits appears around $\mathcal{F}_{L_3}$ when $e_0 > 0.917$. In opposition to QS, this new domain seems satellized around the Ptolemea equant point: we suggest to call it the equant satellite domain.

References:

Jackson,J. 1913. Retrograde satellite orbits. *Mon. Not. R. Astron. Soc.* **74**, 62-82.

Henon, M. 1969. Numerical exploration of the restricted problem. VI. Hill's case: Periodic orbits and their stability. *Astro. Astrophys.* **9**, 24-36.

Nesvorny, D., Thomas, F., Ferraz-Mello, S., Morbidelli, A. 2002. A perturbative treatment of the co-orbital motion. *Celest. Mech. Dyn. Astron.* **82**, 323-361