An Arnold diffusion mechanism for the Galileo satellites

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1 Introduction

The proliferation of space debris orbiting Earth has stimulated the investigations on their dynamical environments and, especially, on the effect of the small disturbing forces that act on their long-term dynamics (see, e.g., [1], [2]). Among them, the third-body perturbation of the Sun and the Moon generate complex resonant structures that provide non trivial behaviors and chaotic motion (see, e.g.,[3]). As a matter of fact, these resonances, together with the ones associated with the solar radiation pressure, organize the distribution of space debris for high-altitude orbits in the long term in the same way as mean-motion resonances create instabilities in the Solar system. It is enough to mention the so-called Kirkwood gaps of the Asteroid Belt, located exactly at mean-motion resonances of low order with Jupiter (see, e.g, [4]). In-depth investigations of resonances represent an effective mean to mitigate the space debris problem (see, e.g., [5]). Their understanding may provide natural mechanisms that allow to control the long-term dynamics of satellites and, therefore, manage the space traffic. In this framework, we will discuss about the dynamics of a test-particle in the $2g + h$ resonance, where g is the argument of perigee and h the longitude of the ascending node of its orbit.

The first part of the talk, sketched in Sect. 2, will be dedicated to formulate the problem and present a perturbative scheme that leads to an integrable Hamiltonian which gives a complete understanding of the resonant dynamics. Then, recalling that the Galileo constellation orbits the Earth in a small neighborhood of the $2g + h$ resonance, we will discuss about the topology of the phase space in the case of a Galileo navigation satellite (see Sect. 3). Finally, going back to the full problem, we will outline our strategy step by step in order to obtain a rigorous proof of Arnold diffusion in the considered problem (see Sect. 4). As proposed by Daquin et al.[6], this mechanism of diffusion may provide a practical application in order to manage the end-of-life of the Galileo constellation by pumping-up the eccentricity of the orbit and slowly guiding the satellites to a reentry in the Earth's atmosphere.

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2 Modeling the problem

We consider the dynamics of a test particle whose Kepler motion around Earth is disturbed by the secular and quadrupolar approximations of the geopotential (usually known as the J_2 effect) and of the third-body perturbation due to the Moon. The Delaunay action-angle variables are introduced in order to preserve the symplectic geometry of the problem:

$$\begin{array}{lll} L = \sqrt{\mu a}, & G = L\sqrt{1-e^2}, & H = G\cos I, \\ l = M, & g = \omega, & h = \Omega, \end{array}$$

where μ is the mass parameter of the Earth and $(a, e, I, \Omega, \omega, M)$ denote respectively the semi-major axis, the eccentricity, the inclination with respect to the equatorial plane, the longitude of the node, the argument of the perigee, and the mean anomaly of the particle.

The problem is approached through the perturbation theory. For that purpose, we define the small parameter $\alpha = a/a_{\mathrm{M}}$ which characterizes the distance of the Moon (a_{M} denotes the semi-major axis of the Moon) with respect to the orbit of the satellite. In that framework, the Hamiltonian of the problem can be written

$$\mathrm{H_K}(L) + \mathrm{H_0}(L, G, H) + \alpha^3 \mathrm{H}_1^{I_{\mathrm{M}}}(L, G, H, g, h, t).$$

where $\mathrm{H_K}$ is the unperturbed Kepler motion of the particle, while $\mathrm{H_0}$ and $\mathrm{H}_1^{I_{\mathrm{M}}}$ model respectively the variations generated by the Earth's J_2 effect and by the Moon. For the sake of conciseness, the readers are referred to the paper [7] for the detailed expressions of each term. We only point out that the Moon's disturbing effect depends on time since its orbit is inclined with respect to the ecliptic ($I_{\mathrm{M}} \simeq 5^\circ$), and experiences a linear drift in the longitude of the node, with a period of about 18.6 years.

2.1 The "$2g + h$" resonance

Our work focuses on a peculiar region of the phase space for which the solutions are characterized by a resonant angle $x = 2g + h$ that oscillates around a given value. At first order, the unperturbed Hamiltonian ($\alpha = 0$), which is integrable, reveals the location of the resonance in the phase space, that is for all $G \neq 0$ and $H = G \cos I_\star$ with a critical inclination $I_\star \simeq 56.06^\circ$ in the prograde case[1]. Besides, in a small enough neighborhood of $I_\star$, the angular variables evolve at different rates, g and h being "fast" angles with respect to x which undergoes a "slow" drift in $\mathcal{O}(|I - I_\star|)$. In the full problem ($\alpha > 0$), the phase space is no more integrable, however the timescales separation still remains as long as $|I - I_\star|$ and α are small enough. A classical way to take advantage of this feature consists in introducing a suitable set of action-angle variables. We propose the symplectic transformation

$$\Upsilon : (l, L, x, y, h, \Gamma, t) \mapsto (l, L, g, G, h, H, t)$$

such as

$$y = \frac{\sqrt{\mu a}}{2}\sqrt{1-e^2}, \quad \Gamma = \frac{\sqrt{\mu a}}{2}\sqrt{1-e^2}\,(2\cos I - 1)\,,$$

in order to deal with a resonant action y that only depends on the variations of the eccentricity. Hence, the variations of the inclination are deduced from Γ.

2.2 A suitable perturbative treatment

The Hamiltonian of the full problem is time-dependent due to motion of the Moon's node with respect to the ecliptic. In order to overcome this difficulty, a first reduction is performed by considering the orbit of the Moon in the ecliptic plane, that is, for $I_{\mathrm{M}} = 0$, which makes the longitude of the ascending node not defined. In that framework, the full problem becomes the perturbation of an autonomous Hamiltonian by a remainder in $\mathcal{O}(\alpha^3 I_{\mathrm{M}})$.

The autonomous Hamiltonian has 3 degrees of freedom, with a conserved quantity L. Another reduction is possible by exploiting the timescales separation and replacing the original Hamiltonian by another one in which the fast oscillations have been removed. In other words, we perform an averaging of the autonomous Hamiltonian over the the fast angle h. According to the perturbation theory, the autonomous Hamiltonian is mapped to the averaged one added to a remainder in $\mathcal{O}(\alpha^6)$.

As a consequence, the two steps of reduction provide an averaged Hamiltonian

$$\mathrm{H_K} + \mathrm{H_0} \circ \Upsilon + \alpha^3 \int_0^{2\pi} \mathrm{H}_1^0 \circ \Upsilon \mathrm{d}h,$$

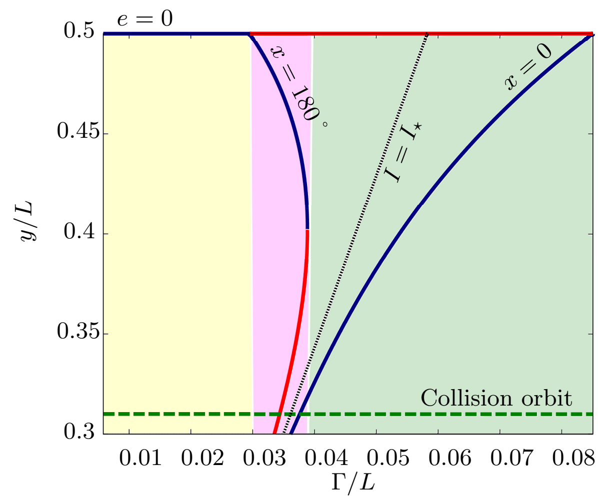

Figure 1: Location of the three families of fixed points, denoted "$e = 0$", "$x = 0$" and "$x = 180^\circ$" in the dimensionless action space $(\Gamma/L, y/L)$. Blue and red curves correspond respectively to segment of centers and saddles. Green, purple and yellow areas are associated with 3 different topologies of phase portraits.

that only depends on (L, x, y, Γ) and for which L and Γ are first integrals. Considering L and Γ as parameters, the description of the phase portrait obtained for various values of Γ allows to understand the global dynamics of the $2g + h$ resonance.

3 The resonant dynamics of a Galileo satellite

From now on, we consider a Galileo satellite that orbits Earth at $a = 29600$ km, that is, for a small parameter $\alpha = 0.077$. The explicit expressions of $\mathrm{H_0}$ and H_1^0 given in [7] combined with our perturbative scheme provide a family of integrable Hamiltonian, parametrized by Γ, that can be written as follow:

$$\mathcal{H}^\Gamma(x, y) = y^{-5}(A + \alpha^3 \sqrt{B} C \cos x)$$

where A, B and C are polynomial functions in (Γ, y).

For each value of Γ, the derived equations of motion allow to compute fixed points that necessarily satisfy one of the following conditions:

$$e = 0, \;\; e > 0 \text{ with } x = 0, \; e > 0 \text{ with } x = 180^\circ.$$

For each condition, a one-parameter family of fixed points is highlighted while the Hessian matrix of the Hamiltonian provides the evolution of its stability. We point out that the resonant variables (x, y), derived from the Delaunay coordinates, have an important failing, that prevents from computing the stability of the fixed points associated with the circular orbit. This difficulty is overcome with the introduction of canonical polar coordinates

$$(\xi, \eta) = \sqrt{2L - 4y}\,(\cos(x/2), \sin(x/2))$$

[1] The retrograde case located in $I_\star \simeq 110.99^\circ$ will not be considered during the talk.

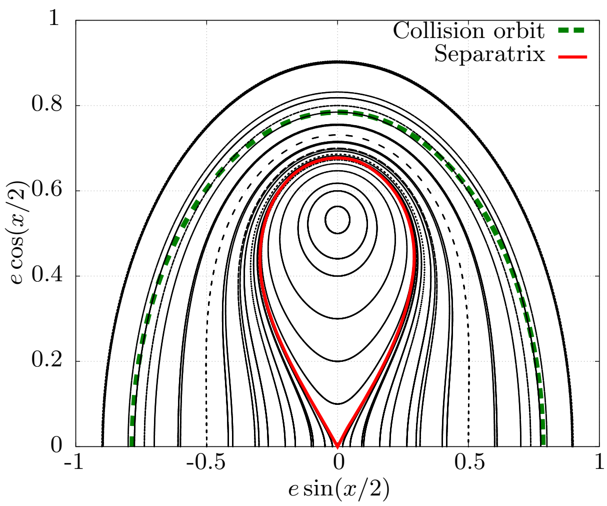

Figure 2: Phase portrait of the averaged Hamiltonian for a Galileo satellite ($\alpha = 0.077$) with $\Gamma/L = \cos I - 1/2$ and $I = 56^\circ$.

that are equivalent to $(e\cos(x/2), e\sin(x/2))$ for quasi-circular orbits.

Figure 1 depicts the location and stability of the three families of fixed points in the dimensionless action space $(\Gamma/L, y/L)$. The families "$x = 0$" and "$x = 180^\circ$" extend from either side of the critical inclination. By varying Γ, "$x = 0$" remains a center, while "$x = 180^\circ$" and "$e = 0$" bifurcate. Hence, by varying Γ, three topologies of phase portraits can be identified: a saddle and a center, respectively in $e = 0$ and $x = 0$ in the green region, two saddles and two centers, respectively in $e = 0$, $x = 180^\circ$ and $x = 0$, $x = 180^\circ$ in the purple region, and a saddle and two centers, respectively in $x = 180^\circ$, $e = 0$ and $x = 0$ in the yellow region. As a consequence, as depicted in the phase portrait of Fig. 2, for a Galileo satellite in circular orbit with $I = 56^\circ$ (green region), a small departure in eccentricity will necessarily lead to a slow increasing of the eccentricity that can reach high values, comparable to the one associated with the collision orbit with the Earth's surface ($e \simeq 0.78$). Two dynamics are possible: a resonant motion inside the separatrix with x and e that oscillate respectively around 0° and $e \simeq 0.55$ with large amplitudes, and a non-resonant motion with x that circulates.

4 A strategy to prove Arnold diffusion

Our aim is to obtain a rigorous proof of existence of a drift in actions, that may increase the eccentricity of a Galileo satellite in the full problem. A normally hyperbolic invariant manifold (NHIM), that has stable and unstable invariant manifolds will be a key tool to construct the drifting orbits.

The ideal case given by our integrable approximation of the problem provides a global understanding of the "$2g + h$" resonance. More precisely, for a fixed value of energy, the dynamics is foliated by two dimensional invariant tori with constant Γ (either resonant or non-resonant depending on the value of Γ). For each value of Γ belonging to a given non empty range $[\Gamma_-, \Gamma_+]$, the averaged Hamiltonian has a saddle in $\xi = \eta = 0$, and the union of these saddles forms a NHIM.

Going back to the non-averaged problem defined by the Hamiltonian with $I_{\mathrm{M}} = 0$, Γ is not integrable any more but the energy is still preserved. In physical terms, the eccentricity cannot increase significantly. In that framework, we will show that each saddle becomes now a hyperbolic periodic orbit implying that the considered NHIM is foliated by invariant two dimensional tori.

In the full problem given by $I_{\mathrm{M}} > 0$, the time dependence due to the motion of the Moon's node is added and the energy is not a first integral anymore. The dynamics in the NHIM will be more complicated, but expected to be I_{M}-close to integrable (we recall that $I_{\mathrm{M}} \simeq 5^\circ$). In such a case, the homoclinic structures constructed in the previous steps will be used in order to obtain Arnold diffusion orbits. Through this strategy, we will build orbits that travel along the invariant manifolds and undergo an increase of energy corresponding to a drift in eccentricity.

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